EULER MAP PROJECTION FOR
THE REPRESENTATION OF EUROPE

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EULER VETÜLETE EURÓPA ÁBRÁZOLÁSÁRA

Összefoglalás

Az Európa területtartó ábrázolására készült térképek gyakran készülnek kúpvetületben. A szögtorzulások szempontjából hagyományosan az Albers vetületet tekintik legjobbnak. A szögtorzulások csökkentése céljából Euler felvetette a területtartó és egyszersmind merőleges fokhálózatú vetület gondolatát. A területtartó henger-, sík- és kúpvetületeken kívül végtelen sok Euler-vetület létezik; ezek közül közelítőleg meghatároztuk azt, amelynek az átlagos szögtorzulása a legkisebb. A maximális szögtorzulás értékei kisebbek e vetületben, mint az Albers-vetületben, és a szögtorzulások egyenletesebben oszlanak el. Ezért előnyös torzulású vetületként ajánlható Európa térképeihez.

Summary

Maps for the equal-area representation of Europe are often made in conical projections. From the point of view of angular distortions, the Albers projection is traditionally considered best. In order to decrease the angular distortions, Euler suggested the use of an equal-area projection with orthogonal graticule. In addition to cylindrical, azimuthal and conical equal-area projections, there are innumerable Euler projections; among these, approximately the one with minimal mean angular error was constructed. The maximal angular distortion values are less in this projection than in the Albers one, and the angular distortions distribute more uniformly. So it can be recommended as favourable for maps of Europe.

The representation of Europe on one map sheet can be originated in the maps of Ptolemy. Similarly, the equal-area (equivalent) projections are rooted in the work of Ptolemy; however, they occur more often among the projections of geocartography only from the end of the 18th century. Nowadays, the chorographical maps of Europe are frequently made in equal-area projections, and depending on the topic, the thematic maps often use equal-area projections, too.

The equivalency of the map affects the increasing of the angular distortions, which causes the torsion of the shape of the continents. Euler (1777) aimed to reduce the angular distortions as a consequence of the equivalency by orthogonal graticule, that is his ambition was constructing projections, beeing simultaneously equal-area and orthogonal.

Be a point given by the geographical coordinates β (the complementary angle of the latitude φ: $\beta = 90° - \varphi$) and the longitude λ, and by the map coordinates x, y . Then the Euler projections are described by the following two differential equations:

$$\tau = \frac{\dfrac{dx}{d\lambda}\cdot\dfrac{dy}{d\beta} - \dfrac{dx}{d\beta}\cdot\dfrac{dy}{d\lambda}}{\sin \beta} = 1 \qquad \text{(equivalency)}$$

$$\operatorname{ctg} \theta = \frac{\dfrac{dx}{d\lambda}\cdot\dfrac{dx}{d\beta} + \dfrac{dy}{d\beta}\cdot\dfrac{dy}{d\lambda}}{\dfrac{dx}{d\lambda}\cdot\dfrac{dy}{d\beta} - \dfrac{dx}{d\beta}\cdot\dfrac{dy}{d\lambda}} = 0 \qquad \text{(orthogonality of the graticule)}$$

where τ is the area distortion, and θ is the angle between the parallels and meridians on the map.

This system of partial differential equations can be solved; namely, the normal Lambert azimuthal, cylindrical and equal-area conical projections, furthermore the Albers conical projection satisfy all the two conditions. In addition to these trivial solutions, innumerable solutions exist. Urmayev (Urmayev 1947) and Meshcheryakov (Meshcheryakov 1968) studied the Euler projections. The latter gave a numerical solution for the hemisphere. The question is how to use the Euler projections for representing Europe with favourable distortions.

The area of Europe can be approximately covered by a geographical quadrangle F, bordered by the latitudes $\varphi_1 = 55°N$ and $\varphi_2 = 20°N$ ($\beta_1 = 35°$ and $\beta_2 = 70°$) and the longitudes ($\lambda_1 = 10°W$ and $\lambda_2 = 60°E$). This area is located away both from the Equator and the Northpole, therefore it is not expected to be represented with advantageous distortions in cylindrical or azimuthal projection. Among the conical ones, the Albers projection with two standard parallels is prospectively more favourable than the equal-area Lambert projection with one true scale parallel.

To find a right particular solution satisfying the above system of differential equations, an additional condition is needed. As the idea is to reduce the angle distortions, characterized by the mean angular error E^2_{aK} , let us average the values of the Kavrayskiy angular error ε^2_{aK} on the geographical quadrangle F, where:

$$\varepsilon^2_{aK} = Ln^2\left(\frac{a}{b}\right)$$

(a and b are the maximum and minimum scale values in the points of F. They can be calculated by the partial differencial of the mapping equations.) So:

$$E^2_{aK} = \frac{1}{\mu(F)}\cdot\int_F \varepsilon^2_{aK}\, dF = \frac{1}{(\cos \beta_2 - \cos \beta_1)\cdot(\lambda_2 - \lambda_1)}\cdot\int_{\beta_1}^{\beta_2} Ln^2\left(\frac{a}{b}\right)\cdot\sin \beta\, d\beta \to \min$$

Due to the location of the area to be represented, it is advantageous if the mapping equations are written for every parallel in polar form (*Figure 1*):

$$x = \rho \cdot \sin \gamma$$
$$y = c - \rho \cdot \cos \gamma$$

where $\rho = \rho(\beta,\lambda)$: the polar distance of the point β,λ;

$\gamma = \gamma(\beta,\lambda)$: the polar angle;

$c = c(\beta)$: the distance of the polar origo of the parallel β from the rectangular origo.

In the first step, the distortions on the graticule (the scale h along the parallels, the scale k along the meridians, and the angle θ) will be calculated:

$$h = \frac{\sqrt{\rho^2 \cdot \left(\frac{\partial \gamma}{\partial \lambda}\right)^2 + \left(\frac{\partial \rho}{\partial \lambda}\right)^2}}{\sin \beta}$$

$$ctg\,\theta = \frac{-\rho \cdot \frac{\partial \gamma}{\partial \lambda}\left(\rho \cdot \frac{\partial \gamma}{\partial \beta} + \frac{dc}{d\beta} \cdot \sin \gamma\right) - \frac{\partial \rho}{\partial \lambda}\left(\frac{\partial \rho}{\partial \beta} - \frac{dc}{d\beta} \cdot \cos \gamma\right)}{\rho \cdot \frac{\partial \gamma}{\partial \lambda}\left(\frac{\partial \rho}{\partial \beta} - \frac{dc}{d\beta} \cdot \cos \gamma\right) - \frac{\partial \rho}{\partial \lambda}\left(\rho \cdot \frac{\partial \gamma}{\partial \beta} + \frac{dc}{d\beta} \cdot \sin \gamma\right)}$$

$$k = \frac{\rho \cdot \frac{\partial \gamma}{\partial \lambda}\left(\frac{\partial \rho}{\partial \beta} - \frac{dc}{d\beta} \cdot \cos \gamma\right) - \frac{\partial \rho}{\partial \lambda}\left(\rho \cdot \frac{\partial \gamma}{\partial \beta} + \frac{dc}{d\beta} \cdot \sin \gamma\right)}{\sin \theta \cdot \sqrt{\rho^2 \cdot \left(\frac{\partial \gamma}{\partial \lambda}\right)^2 + \left(\frac{\partial \rho}{\partial \lambda}\right)^2}}$$

From these follow a and b:

$$a = \frac{\sqrt{h^2 + k^2 + 2 \cdot h \cdot k \cdot \sin \theta} + \sqrt{h^2 + k^2 - 2 \cdot h \cdot k \cdot \sin \theta}}{2}$$

$$b = \frac{\sqrt{h^2 + k^2 + 2 \cdot h \cdot k \cdot \sin \theta} - \sqrt{h^2 + k^2 - 2 \cdot h \cdot k \cdot \sin \theta}}{2}$$

From a and b ε^2_{aK} can be calculated, and in view of the values of ε^2_{aK} the mean error E^2_{aK} can be calculated by the two-dimensional Simpsons form.

The startup system of equations of the Euler projections in this case will be:

$$\tau = \frac{\rho \cdot \frac{\partial \gamma}{\partial \lambda}\left(\frac{\partial \rho}{\partial \beta} - \frac{dc}{d\beta} \cdot \cos \gamma\right) - \frac{\partial \rho}{\partial \lambda}\left(\rho \cdot \frac{\partial \gamma}{\partial \beta} + \frac{dc}{d\beta} \cdot \sin \gamma\right)}{\sin \beta} = 1 \qquad \text{(equivalency)}$$

$$ctg\,\theta = \frac{-\rho \cdot \frac{\partial \gamma}{\partial \lambda}\left(\rho \cdot \frac{\partial \gamma}{\partial \beta} + \frac{dc}{d\beta} \cdot \sin \gamma\right) - \frac{\partial \rho}{\partial \lambda}\left(\frac{\partial \rho}{\partial \beta} - \frac{dc}{d\beta} \cdot \cos \gamma\right)}{\tau \cdot \sin \beta} = 0 \qquad \text{(orthogonality)}$$

To the approximative solution of the equation system completed with minimum condition, the functions $\rho = \rho(\beta,\lambda)$, a $\gamma = \gamma(\beta,\lambda)$ and $c = c(\beta,\lambda)$ will be constructed in polynomial form:

$$\rho(\beta,\lambda) = \rho_{11} \cdot \beta + \rho_{12} \cdot \beta^2 + \lambda^2 \cdot \left(\rho_{21} \cdot \beta + \rho_{22} \cdot \beta^2\right)$$
$$\gamma(\beta,\lambda) = f_{11} \cdot \lambda + f_{12} \cdot \lambda^3 + \beta \cdot \left(f_{21} \cdot \lambda + f_{22} \cdot \lambda^3\right)$$
$$c(\beta) = t_1 \cdot \beta + t_2 \cdot \beta^2 + \rho(\beta,\lambda) = t_1 \cdot \beta + t_2 \cdot \beta^2 + \rho_{11} \cdot \beta + \rho_{12} \cdot \beta^2 + \lambda^2 \cdot \left(\rho_{21} \cdot \beta + \rho_{22} \cdot \beta^2\right)$$

The polynomial coefficients ρ_{11}, ρ_{12}, ρ_{21}, ρ_{22}, f_{11}, f_{12}, f_{21}, f_{22}, t_1, t_2 were approximated by downhill simplex method. The results are:

$\rho_{11} = 1.014909$, $\rho_{12} = 0.179632$, $\rho_{21} = -0.019402$ $\rho_{22} = 0.028744$
$f_{11} = 1.001009$, $f_{12} = -0.007598$, $f_{21} = -0.269217$, $f_{22} = -0.021786$
$t_1 = -1.032131$, $t_2 = 0.030571$

These coefficients help us to satisfy approximately the conditions of equivalency and orthogonality on the geographical quadrangle F; furthermore, the mean angular error on F is minimal.

The angular distortions of the enumerated Euler projections on the geographical quadrangle F were examined in the following respects: the maximum $2{\cdot}\omega_{max}$ of the maximal angular deformations $2{\cdot}\omega = 2{\cdot}\arcsin((a-b)/(a+b))$ were calculated as well as the maximum $\varepsilon^2_{aK\ max}$ and the mean E^2_{aK} of the Kavrayskiy angular error $\varepsilon^2_{aK} = \mathrm{Ln}^2(a/b)$. The results are summarized in the decreasing sequence of the means E^2_{aK} in *Table 1*:

Table 1

Name of projection	$2{\cdot}\omega_{max}$	$\varepsilon^2_{aK\ max}$	E^2_{aK}
Lambert cylindrical projection	104.47°	4.604	1.148
Best equal-area cylindrical projection for F ($\varphi_n = \pm52°$)	63.78°	1.38	0.209
Lambert equal-area azimuthal projection	13.71°	0.0575	0.0209
Lambert equal-area conical projection ($\beta_n = 41°$, that is $\varphi_n = 49°$N)	6.24°	0.0114	0.00364
Albers equal-area conical projection ($\beta_1 = 46°$, that is $\varphi_1 = 44°$N; $\beta_2 = 26°$, that is $\varphi_2 = 64°$N)	4.71°	0.0068	0.000790
Euler projection with minimal mean angular error for Europe	3.47°	0.0037	0.000741

Let's look at the optimal Albers projection (having standard parallels: $\beta_1 = 46°$, that is $\varphi_1 = 44°$N and $\beta_2 = 26°$, that is $\varphi_2 = 64°$N; *Figure 2*) and the approximated Euler projection with minimal angular error for F (*Figure 3*), and we compare their angular distortions, namely the values of their maximum angular deformation 2ω and the Kavrayskiy angular error ε^2_{aK}. You can see the result in the figures showing the isocols (the lines of equal distortion) of these values.

The isocols in *Figure 2* are concentric circles, going on along the parallels. The minimal angular distortion appears on the standard parallels, the maximal one on parallel 70°N. The direction of the isocols in *Figure 3* converges to the lower bordering parallel and lateral bordering meridians of F. The angular distortions distribute more uniformly. The minimal values are located on the central meridian (25°E), on latitude 59°N. The maximum values are on the 68°N latitude, on the brink of F. The meridian convergency on the lower corners of F is less in the Euler projection, too.

Summary of the results by means of the above comparison: the angular distortion is more favourable in the approximated Euler projection with minimal mean angular error than in the Albers projection traditionally considered best. On this basis, the author recommends this projection for the equal-area representation of Europe or for similar purposes.
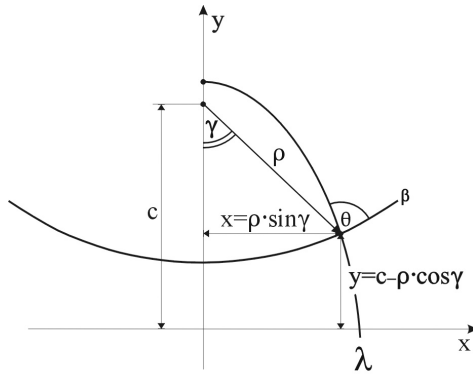
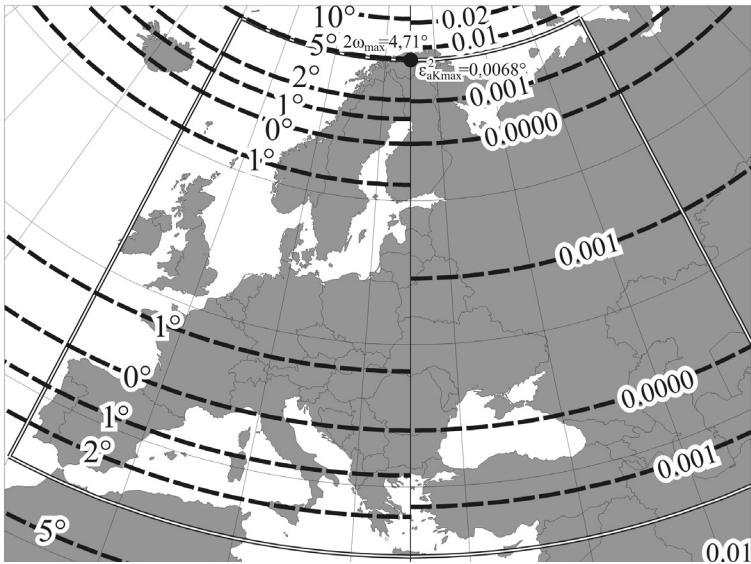*Figure 1. The construction of a projection by polar coordinates*

*Figure 2. Albers equivalent conical projection: the isocols of the maximum
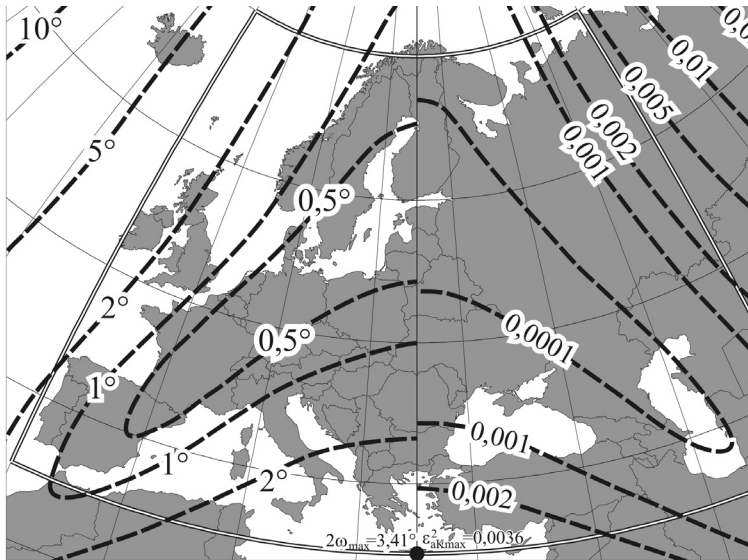angular deformation and the Kavrayskiy angular criterion*

Figure 3. Approximated Euler projection with minimal angular error: the isocols of the maximum angular deformation and the Kavrayskiy angular criterion

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